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BROUWER-HEYTING MONOIDS

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ABSTRACT

Notation of BH monoids and BH lattices are introduced and decomposition theorem are obtained.

KEYWORDS: Brouwer-Heyting monoids (BH monoids), BH lattice, residuation.

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1. INTRODUCTION

A lattice L (L.V, ∧)I which to each a,b there is a largest x such that x∧b ≤a is called a Brouwerian lattice ([1]).But these are also called Heyting algebras. But in [3], Brouwerian lattice(algebra or logic) is a lattice in which to each a,b there is a least x such that xVb≥a.

Ever since ward and Dilwarth ([6] initialed the study of residulated lattices, lot of interest is being shown in the study of lattice ordered monoids with residuation as an adjoint operation .Swamy ([5]) initiated the study of dually residualed lattice ordered semigroups (DRL. Semigroup) in the context of obtaining a common abstraction of Brouwerian algebras (of [1]) and lattice ordered groups. They are called dually residuated because the residuation on is dual to that of ward and Dilworth.

In this paper ,we study structures (P,a,e,ρ,→) wher (,ρ ,a,e) is a commoutative monoid,in P,where ,ρ is a partial ordering on P,and → standsfor binary operation such that for all x,a,b in P (xob) ,ρ a ⇔ x,ρ (a→ b).

If o is the ∧ operation and ,ρ = ≤, then the inclusion → coincides with that of Browerian algebras (of [1]).If o is the V operation and ,ρ = ≥, then ,this residuation →coincides with that of Browerian algebras of BCK monoids (commutative) of Subrahmanyam([4]) and DRL monoids of swamy ([5]) can be realised as duals of these systems once when ,ρ is fixed. We call these systems as Brower-Heyting monoids (BH-monoids).Bounded BH monoids which are lattices are commutative residuated lattice ordered monoids studied by U.Hoble ([2]).

We call BH monoids which are lattices with a certain divisibility condition as BH lattices.They are duils of DRL monoids.

The main results of the paper are decomposition theorem for BH monoids and BH lattices. we obtain a necessary and sufficient condition for a BH monoid to decompose as the direct product of commutative po-group and a BH monoid with greatest element (Theorem 4.1).In theorem 4.2,we establish that any BH lattice is the direct product of a commutative l-group and a BH lattice with greatest element .Section (1) and (2) deal with definitions, examples and preliminary algebra of BH monoids, section (3) deals with BH lattices and section (4) contains the decomposition theorems.

2. BH MONOIDS

We begin with the following definition

Definition1.1: A system (G,o,e, ,ρ ,→) where (i).(G.o,e) is a commutative semi group with identity "e" (ii) (G, ,ρ) is a partially ordered set and → is binary operation on G.Such that ,for all x,a,b in G (xob) ρ a ⇔ x,ρ (a→ b).
We now the give some examples.

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Example1.1 (G,o,e,, ,ρ) is a commutative po-group. Let a→b=aob^{-1}.

Example 1.2 (B,∪,∩,' 0,1)is a Boolean algebra. Let o=∩, ,ρ =≤ defined by a,ρ b if
a∩b=a, e=1, a→b = a∪ b^1.

Examples 1.3 Let (G, ∪,∩,1)be a Heyting algebra.By definition in G,given a,b there is a largest x such that b∩x ≤ a. Let o = ∩ e = 1 ,ρ is the lattice order ,a→b is the x described above.

Examples 1.4 Let (L,V,Λ,O) be Brouwerian lattice ([3]).It is a lattice with "0" in which given a,b there is a smallest x (denoted by a-b) such that xVb ≥ a i.e .a ≤ xVb ⇔ a-b ≤ x. If we take ,ρ=(the dual of ≤) ≥,then this condition reads: (xVb) ,ρ a ⇔ x,ρ (a-b). Thus a Brouwerian lattice with dual ordering is a BH monoid ,with V,as monoid operation and the least 0 as identity. Thus if (L,V,Λ,0,-) is a BH monoid where ,ρ is the dual ordering of the lattice (L,V,Λ).

Examples 1.5 Let (G,+, ≤, -, 0) be a DLD-monoid.We have x+b ≥ a ⇔ a-b ≤ x (By the definition of DRL-monoid). Thus (G,+,≥,-, 0) is BH monoid so the dual of DRL-monoid is BH monoid .

Examples 1.6 Let (G,0,e,≤,:)be a commutative BCK-monoid. The defining condition of a BCK monoid is x≤aob ⇔ x:b≤a. If we define ,ρ = ≥ (the dual of ≤),then the condition reads (aob) ,ρ ⇔ a,ρ (x:b). Thus the dual of a commutative BCK monoid is BH monoid (G,o,e. ,ρ ,:).

3. PRELIMINARY RESULTS

In what follows for the sake of connivance we use ≤ instead of ,ρ. Let (G,o,e,≤,→) be a BH monoid.

Result 2.1 b≤c ⟹ aob≤aoc
Proof: coa=aoc ⟹ coa≤aoc
⇒ c≤(aoc)→a ⟹ b≤aoe→ a ⟹ boa≤ aoc
(by the defining condition of →).

Result 2.2 a≤(aob)→b.Clear as aob≤(aob) we get a≤(aob)→b.

Result 2.3: ($\boldsymbol{a}$ →b)ob≤a (from definition).

Result 2.4: c → (aob)=(c→b)→a=(c→a)→b.

Proof: (c → (aob)oaob ≤ c (from result 2.3)
⇒ (c → aob)oa ≤→c→b
⇒ (c → aob)≤→ (c→b)→a…….(i)

Now(c →b)→ a)$o$aob ≤ (c→b)ob ≤c
⇒ (c → b) →a)≤c→aob…….(ii)
From (i) and (ii) give the result.

Result 2.5: e→e =e
Proof: eoe=e ⟹ e≤e→e=(e→e)o≤e

Result 2.6: a→e=a

Proof: aoe=a ⇔ ⟹ a≤a→e=(a→e)oe≤a.

Result 2.7: e ≤ a→a

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Proof: eoa=a ⟹ e≤e→a.

Result 2.8: a ≤b ⟹ a→c≤b→c.

Proof: co (a→c)≤a≤b ⟹ a→c≤b→c.

Result 2.9: a≤b ⟹ c→b≤c→a

Proof: (c→b)oa ≤ (c→b)ob ≤ c →c→ b≤c→ a.

Result 2.10: b≤a ⟺ e≤a →b.

Proof: b≤a ⟺ eob≤a ⟺ e≤a→ b.

Result 2.11: (a→ b)o(b→ c)≤ a→c.

Proof: co (a→b)o(b→ c) = co(b→c)o(a→ b)≤bo(a→ b)≤a
⟹ (a→ b)o(b→ c)≤a→ c

Result 2.12: If a∨b exists for any a,b then,(coa)∨(aob) exists for any c and co(a∨b)= (coa)∨(cob).

Proof: If (coa)≤u, cob≤u then a≤ u→c,b≤u→ c so that
a∨b≤u→c ⟹ co(a∨b)≤u,then other part is routine.

Result 2.13: If a∨b exists for any a,b then (c→a)∧(c→ b) exist and
c→ (a∨b)= (c →a) ∧(c→ b).

Proof: a≤ a∨b, b≤ a∨b ⟹ c→a∨b≤c→ a, c→a∨b≤c−b so that
c→ a∨b is a lower bound of c→ a and c→ b. Now u≤c→a,
u≤c→b ⟹ uoa≤c,uob≤ c ⟹ a≤c→ u,b≤c→ u.
⇒ a∨b≤c→u ⟹ c →(c→ u)≤c→a∨b. i.e u≤c→a∨b as
u≤c→ (c → u).

Result 2.14: If a∧b exists then, (a →c)∧(b→ c).

Proof: u≤ a →c, u≤ b →c ⟹ uoc≤a,b ⟹ uoc≤a∧b
⇒ u≤a∧b→c,other part is obvious as a∧b→ c is a lower bound of a→c
and b→ c

3. BH LATTICES

Definition 3.1 A BH monoid (L,o,c,≤,→) is called a BH lattice if
(i) (L,≤) is a lattice with glb and lub denoted ∧ and ∨ respectively.
(ii) ao(b∧c)=(aob)∧(aoc) for all a,b.c in L
(iii) ((b →a)∧c)oa= a∧b for all a,b in L.

Theorem 3.2 A lattice (L, ∨,∧,a,e,→) where (L,o.e) is a monoid and → be a binary operation on L,is a BH lattice iff
(i) (y → x)ox≤y
(ii) x∧z→y ≤ x→y
(iii) x≤ xoy→y.for all x,y in L
(iv) ao(b∧c) = (aob)∧(aoc) for all a,b,c in L

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(v) $((b \to a)\wedge e)oa = a\wedge b$, for all a,b in L.

Proof: Clearly (i),(ii),(iii) and (iv) are valid in BH lattice.
Assume (L, ∨,∧,a,e,→) satisfies (i),(ii),(iii),(iv)and (v),for any a,b in L,
$(a \to b)ob \leq a$ by (i) Let $xob \leq a$.Now $x \leq (xob) \to b$ by (iii)
$= a\wedge(xob)) \to b \leq a \to b$,by (ii). Also $z \leq a \to b \Rightarrow zob \leq (a \to b)ob$ (by (iv)) $\leq a$.

Remark 3.3 Theorem 3.3 shows that BH lattices can be defined by means of identities alone, so that they form a variety of algebras. Throughout this section let L, denote a BH lattice.

Theorem 3.4 In BH lattice $a \to a = e$,for all a.

Proof: $e \leq a \to a$. Now put $d = a \to a$.We observe dod=d
$e \leq d \Rightarrow eod \leq dod$, i.e $d \leq dod$.
$e \leq a \to a \Rightarrow eoa \leq (a \to a)oa \leq a$
$\Rightarrow a = (a \to a)oa$
$\Rightarrow d = a \to ((a \to a)oa = (a \to a) \to (a \to a) = d \to d$.

By (iii) in the definition of BH lattice , for any x with $e \leq x$
We have $((e \to x)\wedge e)ox = e$. Now $eod = ((e \to d)\wedge e)odod$
$\Rightarrow d = ((e \to x)\wedge e)od \leq (e \to d)od \leq e$ so that d=e.

Theorem 3.5 Any BH lattice is distributive.

Proof: Enough to show that relative complements are uniquely determined. Let $a \leq x \leq b$, let y,y^1 be such that $x\wedge y = x\wedge y^1 = a$. $x\vee y = x\vee y^1 = b$.
Now $y = y\wedge(y\vee x) = ((y \to y\vee x)\wedge e)o(y\vee x) = ((y \to x)\wedge e)o(y\vee x)$
$= (y\wedge x \to x)o(y\vee x) = ((y^1\wedge x) \to x)o(y^1 ox)$
$= ((y^1 \to x)\wedge e)o(y^1\vee x) = ((y^1 \to y^1)\vee x)\wedge e)o(y^1\vee x) = y^1\wedge(y^1\vee x) = y^1$.

Theorem 3.6 In BH lattice, for any a,b prove $(a\vee b)o(a\wedge b) = aob$

Proof: $(a\wedge b) = (a\wedge b \to b)ob = ((a \to b)\wedge c)ob = (a \to a\vee b)ob$.
$(a\wedge b)o(a\vee b) = (a \to a\vee b)obo(a\vee b) = ((a \to a\vee b)o(a\vee b)ob = aob$.

Corollary : $a = aoe = (a\vee e)o(a\wedge e)$.

Theorem 3.7 Any x with $e \leq x$ is invertible.

Proof: We have $(e \to x)ox = e$ as $e \leq x$, so that x is invertiable.
(and inverse of x is $e \to x$).

Theorem 3.8 For any x, $e\vee x$ is invertible.

Theorem 3.9 If x is invertible then $e \to x$ is inverse of 'x'.

Proof: $yox = e \Rightarrow y \leq e \to x$
$\Rightarrow yox \leq e \to xoa \leq = e \Rightarrow e = (e \to x)\, ox$.

Theorem 3.10 If b is invertible then $a \to b = ao(e \to b)$.

Proof: $ao(e \to b)ob = a$ let $xob \leq a \Rightarrow xobo(c \to b) \leq ao(e \to b)$.
$\Rightarrow xoe \leq ao\,(e \to x)$.

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i.e x ≤ ao(e → b).So that a→ b=ao(e→ b).

Theorem 3.11 For any x, (e → x)is invertiable.

Proof: If a,b are invertible ,then clearly aob is invertible.
Also a→ b=ao(e→ b).Implies that a→ b is also invertiable.
Now x=(x∨e)o(x∧e) ⟹e→ x=(e → x∧e)→x∨e.
x∧e ≤ e⟹e→x∧e ≥e→e = e. So that e→ (x∧e)→(x∨e) is invertible.

Theorem 3.12 If G is the set of all invertible elements then G is a l-group

Proof: If G is already po-group. It is enough to observe that for any x in G ,x∨e is also in G.As any x∨e (≥e) is invertible follows the result.

4. DECOMPOSITION THEOREMS

Theorem 4.1 Decomposition theorem for BH monoids.
A BH monoid L is the direct product of po- group and a BH monoid with greatest element iff

- **(i)** e→ a is invertible for all a in L.
- **(ii)** a → a=e for all a in L.

Proof: Assume (i) and (ii). Let G be the set of all invertible elements of L and
let H = {a ϵ L|e ⟶ a = e}. Clearly (G, ≤, o) is a po-group.
Now e ϵ H so that H is non-empty. Also a ϵH ⇒ e ⟶ a = e ⇒ a ≤ e.
For any a,b in H, e ⟶ (aob) = (e ⟶ a) ⟶ b = e ⟶ b = e.
Let a,b ϵH ⇒ a ⟶ b ≤ e ⟶ b = e ⇒ e ⟶ (a ⟶ b) ≥ e ⟶ b = e.
Now b ≤ e ⇒ a ⟶ b ≥ a ⟶ e = a ⇒ e ⟶ (a ⟶ b) ≤ e ⟶ a = e. So that a ⟶ b ϵ H and hence H is a BH monoid with greatest element.
Now let aϵ L. Put x = ao(e ⟶ a), y = e ⟶ (e ⟶ a). e ⟶ x = e ⟶ ao(e ⟶ a)
= (e ⟶ a)−(e ⟶ a) = e. So that e ⟶ x belongs to H.
y is invertible by hypothesis (i) and so is in G.
Now ao(e ⟶ a)o(e ⟶ (e ⟶ a)) = aoe = a, since (e ⟶ a)) and e ⟶ (e ⟶ a)) are inverses of each other. Now let a = x^1o y^1 , x^1 ϵ H, y^1ϵ G. e ⟶ a = e ⟶ x^1 oy^1 = (e ⟶ x^1)−y^1= e−y^1
Also e ⟶ a = e ⟶ xoy = e−y.
Now e ⟶ y = e ⟶ y^1 ⇒ e ⟶ (e ⟶ y) = e ⟶ (e ⟶ y^1) ⇒ y = y^1 as y and y' are elements of a group.
Now xoy = $x^1$0o y^1 ⇒ xoy = x^1oy ⇒ xoyo(e ⟶ y) = x^1oyo(e ⟶ y) ⇒ x = x^1.
The other part is obvious.

Theorem 4.2: Decomposition theorem for BH lattices
Any BH lattice L is the direct product of a commutative l-group and a BH lattice with greatest element.

Proof: Let G be the set of all invertible elements of L and
let H = {a ϵL|e ⟶ a = e}.

By theorem 4.1, L is already GxH as BH monoids. G is a l-group by theorem3.12. The proof will be complete if we show that H is a BH lattice. For a,b in H, e ⟶ (a∨b) = (e ⟶ a)∧(e ⟶ b) = eoe = e.(By theorem 2.13)
To show e ⟶ (a∧b) = e, e = e ⟶ (aob) = e ⟶ ((a∨b)o(a∧b)) = (e − a∨b) ⟶ (a∧b) = e ⟶ (a∧b). The proof is complete.

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